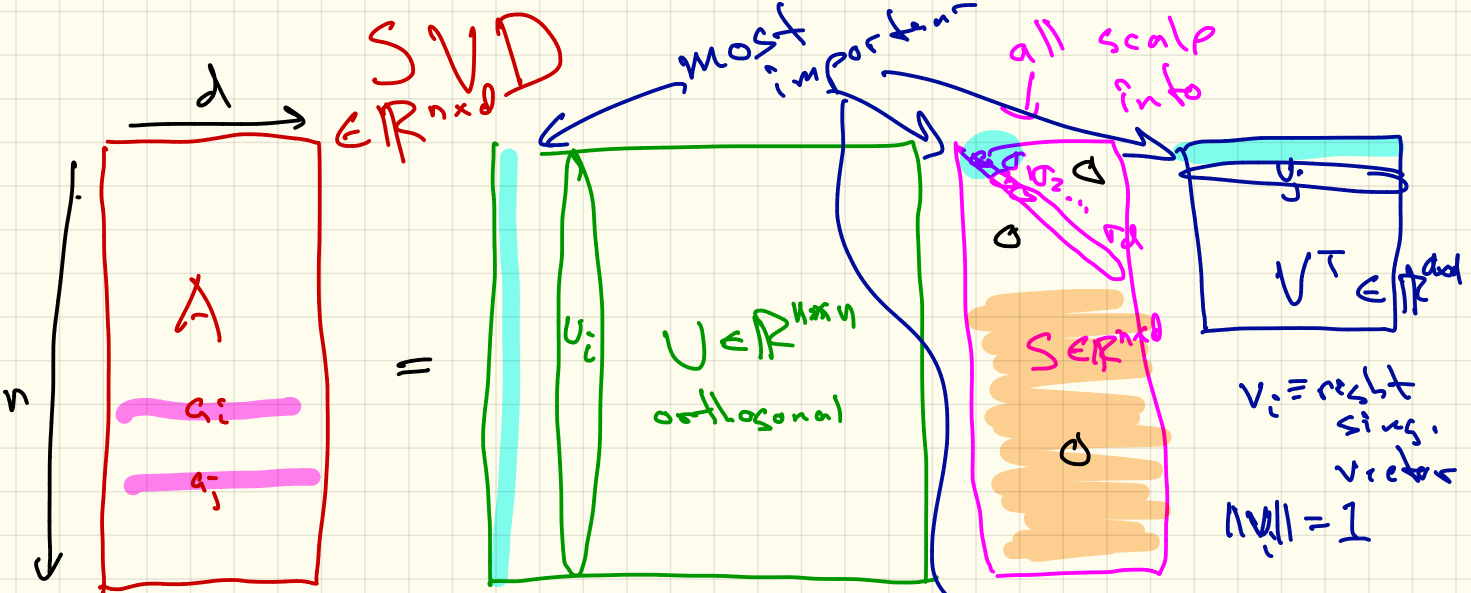
FoDA L19 : Dimensionality Reduction

Rank-k Aproximation & Eigenvalues

SVD



v_i = right sing. vector

$\|v_i\| = 1$

$\langle v_i, v_j \rangle = 0$

$a_i, a_j \in \mathbb{R}^d$

assume

$\|a_i - a_j\| = D(a_i, a_j)$ makes sense

u_i = left singular vectors

$\|u_i\| = 1$

$\langle u_i, u_j \rangle = 0$

singular values

$S_{11} = \sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \sigma_r > 0$

r = rank(A)

orthogonal matrices

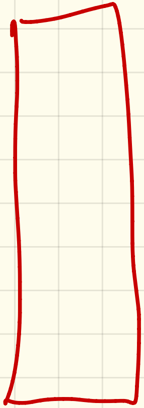
$U^T = U^{-1}$ $\quad UU^T = I$ ⇐ ↳ no scale information

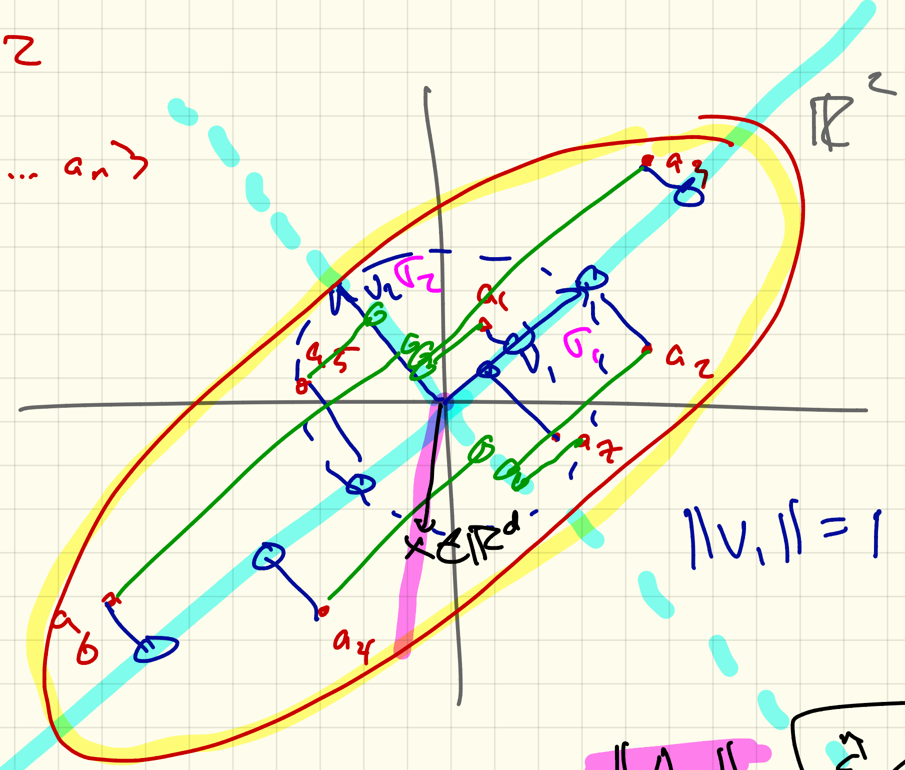
$V^T = V^{-1}$ $\quad VV^T = I$ $\quad \|V^T x\| = \|x\|$

Aggregate Shape

$A \subset \mathbb{R}^2$

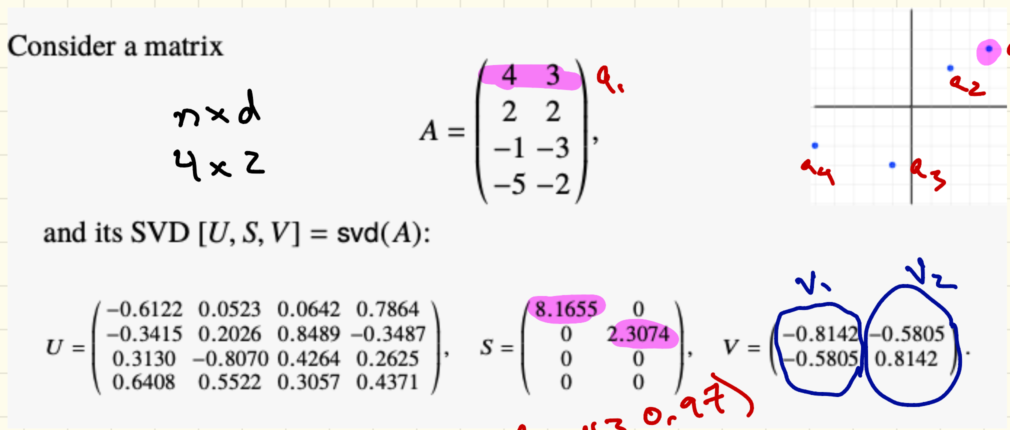
$= \{a_1, a_2 \dots a_n\}$





$\|v_1\| = 1$

$$\|Ax\| = \sqrt{\sum_{i=1}^{n} \langle x, a_i \rangle^2}$$



Consider a matrix

$n \times d$

4×2

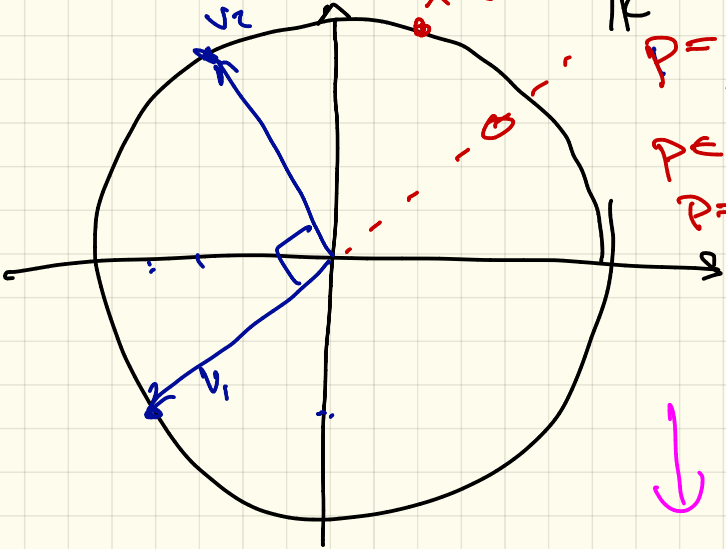
$$A = \begin{pmatrix} 4 & 3 \\ 2 & 2 \\ -1 & -3 \\ -5 & -2 \end{pmatrix}, \quad a_1$$

and its SVD $[U, S, V] = \text{svd}(A)$:

$$U = \begin{pmatrix} -0.6122 & 0.0523 & 0.0642 & 0.7864 \\ -0.3415 & 0.2026 & 0.8489 & -0.3487 \\ 0.3130 & -0.8070 & 0.4264 & 0.2625 \\ 0.6408 & 0.5522 & 0.3057 & 0.4371 \end{pmatrix}, \quad S = \begin{pmatrix} 8.1655 & 0 \\ 0 & 2.3074 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 & v_2 \\ -0.8142 & -0.5805 \\ -0.5805 & 0.8142 \end{pmatrix}.$$

$y = Ax$

$y = U \underbrace{S \underbrace{V^T x}_{p}}_{q}$



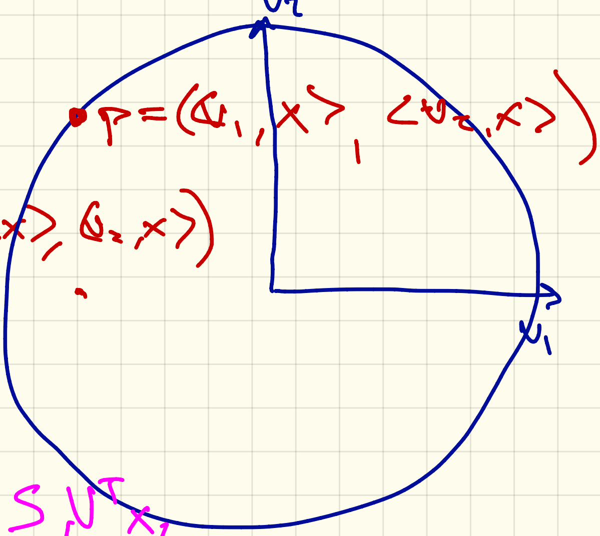
$x = (0.243, 0.97)$

$\mathbb{R}^2$

$p = V^T x$ →

$p \in \mathbb{R}^d$

$p = (\langle v_1, x \rangle, \langle v_2, x \rangle)$

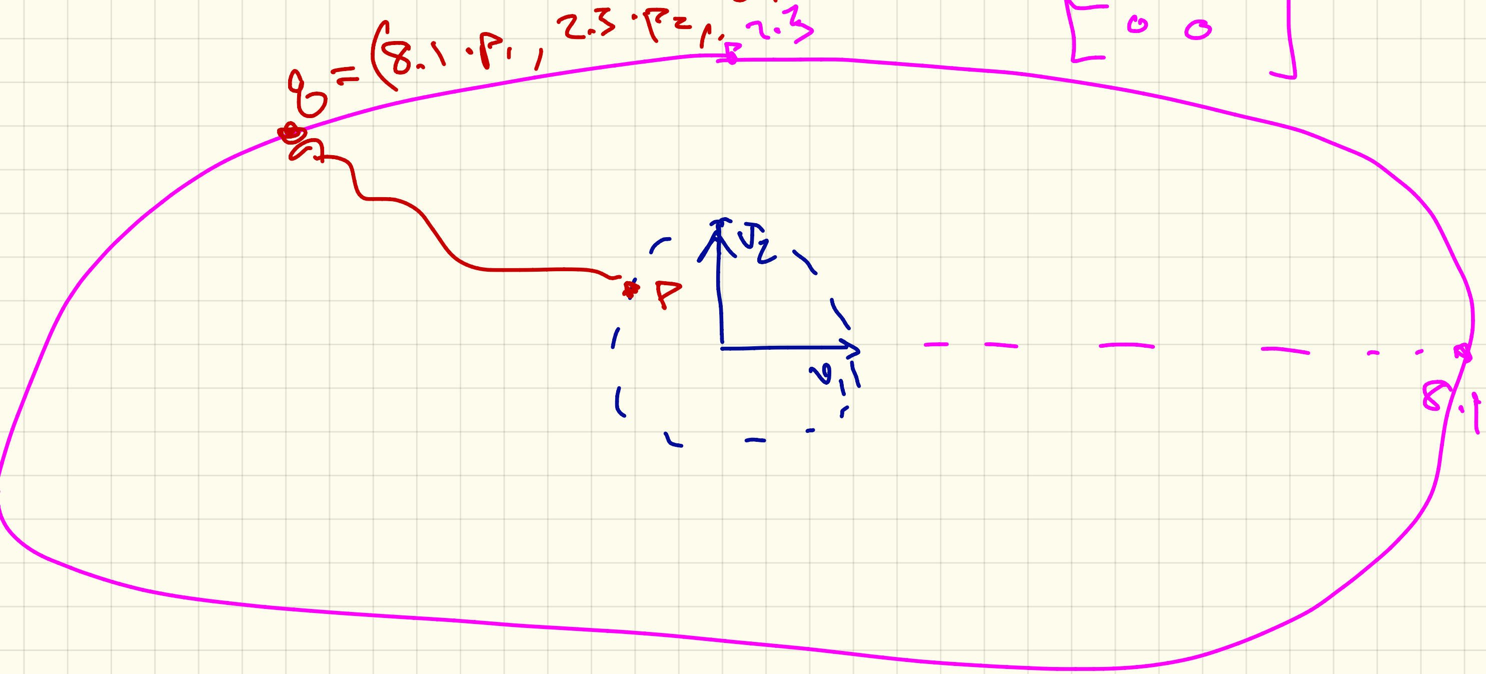


$p = (\langle v_1, x \rangle, \langle v_2, x \rangle)$

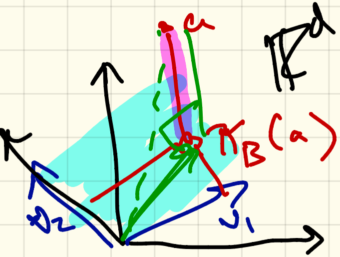
↓ $q = S \underbrace{V^T x}$
$= S p$

$$q = Sp$$

$$S = \begin{bmatrix} 8.1 & 0 \\ 0 & 2.3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$



$$q = (8.1 \cdot p_1 ,\ 2.3 \cdot p_2 ,\ 0 ,\ 0)$$



Best Rank-k Aprox of A

$$B \in \mathbb{R}^{n \times d} \quad \text{so} \quad \text{rank}(B) = k$$

$k < d < n$

to minimize

$$\|A - B\|_2 \quad \text{and/or} \quad \|A - B\|_F$$

top sing. val

$$\sigma_1^2 = \sum_{i=1}^{n} \langle a_i, v_1 \rangle^2 = \max_{\substack{v \in \mathbb{R}^d \\ \|v\|=1}} \sum_{i=1}^{n} \langle a_i, v \rangle^2$$

$$\sigma_j^2 = \sum_{i=1}^{n} \langle a_i, v_j \rangle^2 \quad \Big\| \quad \sigma_2^2 = \max_{\substack{v \in \mathbb{R}^d \\ \|v\|=1 \\ \langle v_2, v_1 \rangle = 0}} \sum_{i=}^{n} \langle a_i, v \rangle^2$$

$$V_B = \{v_1, v_2 \ldots v_k\}$$

$$\pi_B(a) = \sum_{j=1}^{k} v_j \langle a, v_j \rangle$$

$$\|a - \pi_B(a)\|^2 = \Big\| \overbrace{\sum_{j=1}^{d} v_j \langle a, v_j \rangle}^{a} - \overbrace{\sum_{j=1}^{k} v_j \langle a, v_j \rangle}^{\pi_B(a)} \Big\|^2 = \sum_{j=k+1}^{d} \langle a, v_j \rangle^2$$

Find $B \in \mathbb{R}^{n \times d}$ $(\mathrm{rank}(B) = k)$

to minimize $\sum_{i=1}^{n} \| a_i - \pi_B(a_i) \|^2$

$= \| A - B \|_F^2$

$= \sum_{j=k+1}^{d} \sum_{i=1}^{n} \langle a_i, v_j \rangle^2$

$\sum_{j=k+1}^{d} \overbrace{\sum_{i=1}^{n} \langle v_j, a_i \rangle^2}^{\sigma_j} =$ or

$= \sum_{j=k+1}^{d} \sigma_j^2$

to minimize

$\max_{\substack{v \in \mathbb{R}^d \\ \|v\|=1}} \sum_{i=1}^{n} \langle v, a_i - \pi_B(a_i) \rangle^2$

$= \| A - B \|_2^2$

$= \sum_{i=1}^{n} \langle v_{k+1}, a_i \rangle^2$

$= \sigma_{k+1}$

set B : $V_B = \{ v_1, \ldots, v_k \}$

top k right sing vectors

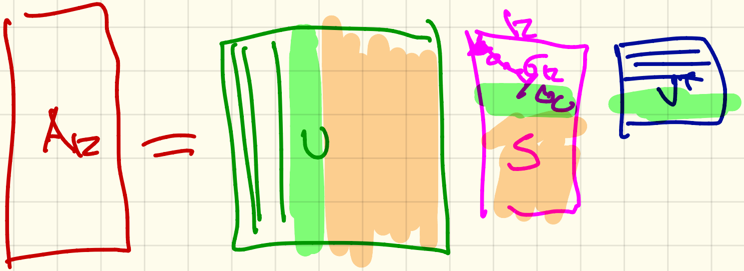
v_1 , σ_1 , u_1

$\sigma_1 u_1 v_1^T \in \mathbb{R}^{n \times d}$ rank 1

$\sigma_j u_j v_j^T \in \mathbb{R}^{n \times d}$ rank 1

$$A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^T \in \mathbb{R}^{n \times d} \quad \text{rank } k$$

↳ minimizes $\| A - A_k \|_F$ and $\| A - A_k \|_2$



Eigenvectors & Eigenvalues

Input $M \in \mathbb{R}^{d \times d}$ ← square

$$Mv = \lambda v$$

eigenvalue λ

v = eigenvector

If M is positive semidefinite

at most n eigen value/vector pairs

all eigen values Real and positive

non-negativ

$$\|v_i\| = 1$$

$$\langle v_i, v_j \rangle = 0$$

$M_R = A^T A \in \mathbb{R}^{d\times d}$ $M_L = A A^T \in \mathbb{R}^{n\times n}$

if A full rank $d < n$

M_R pos. def. M_L pos. semidefin

$U S V^T = A \leftarrow svd(A)$

$$M_R V = A^T A V = (US\underbrace{U^T)(U}_{I}S\underbrace{V^T) V}_{I}$$

$$= V S^2 = V \begin{bmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & \ddots & \sigma_d^2 \end{bmatrix}$$

eigenvector

$v_j = v_j$ right sing. vector

$\lambda_1 = \sigma_1^2$

$\dots$

$\lambda_j = \sigma_j^2$

$$M_L = AA^T \in \mathbb{R}^{n\times n}$$

$$M_L U = AA^T U = (USV^T)(VSU^T)U$$
$$= US^2$$

left sing. vectors

$= $ eigenvectors of $M_L = AA^T$

sing values - squared

$= $ eigen values of $M_L = AA^T$

Eigendecomposition

$$M \in \mathbb{R}^{d \times d}$$

$$M = V L V^{-1}$$

$$= V L V^{T}$$

$V \in \mathbb{R}^{d \times d}$

orthogonal

$L = \text{diag} \in \mathbb{R}^{d \times d}$

$L = \text{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$

↑

eigenvalues

Algo. inverse

$$M^{-1} = (V L V^{T})^{-1}$$

$$= V L^{-1} V^{T}$$

$$L^{-1} = \begin{bmatrix} \lambda_1^{-1} & & & \\ & \lambda_2^{-1} & & \\ & & \ddots & \\ & & & \lambda_d^{-1} \end{bmatrix}$$