FoDA : Linear Algebra
29 Independence + Norms

16 ✓ Yes: Cheat Sheet worth 10 pts

27 ✗ ~~No: Cheat Sheet optional + 5 pts max score still 100~~

Norms

vector norms $\qquad v \in \mathbb{R}^d$

$\|v\|$ "length" $\qquad v = (v_1, v_2, \dots v_d)$

$$\|v\|_2 = \|v\| = \sqrt{\sum_{i=1}^{d} v_i^2}$$

norms for $p \in [1, \infty)$

$$\|v\|_p = \left(\sum_{i=1}^{d} |v_i|^p \right)^{1/p}$$

$v = (1, -6, 3)$

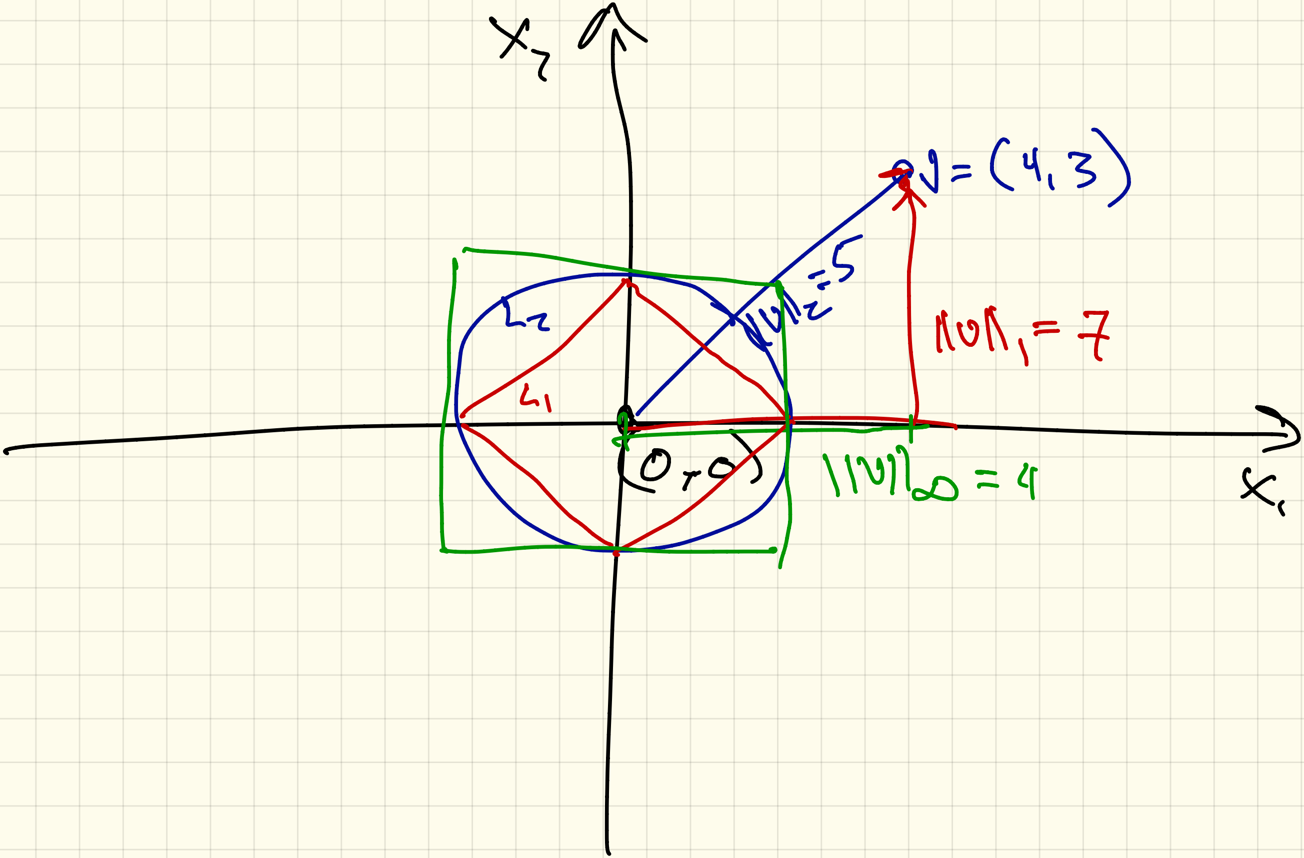
$$\|v\|_1 = \sum_{i=1}^{d} |v_i|$$

$\|v\|_1 = 1 + 6 + 3 = 10$

$\|v\|_2 = \sqrt{1 + 36 + 9} = \sqrt{46} \approx 6.8\ldots$

$$\|v\|_\infty = \max_{i \in [1..d]} |v_i|$$

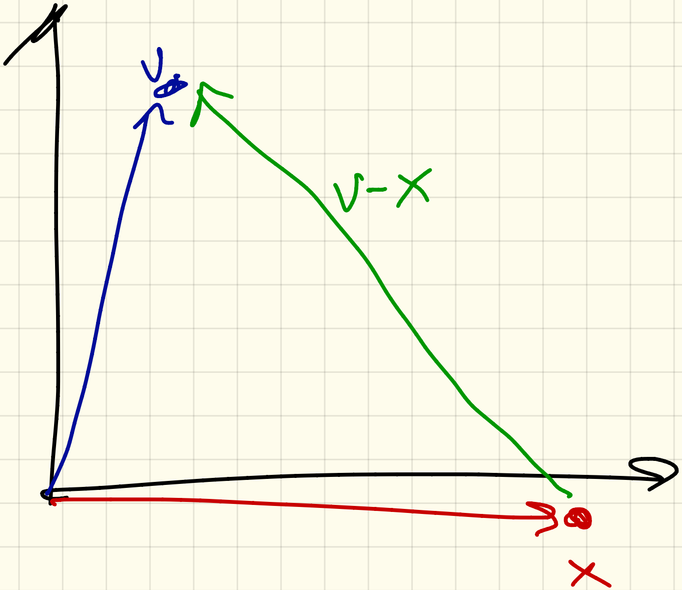
$\|v\|_\infty = 6$



x_2
x_1
$v = (4,3)$
$\|v\|_2 = 5$
$\|v\|_1 = 7$
$\|v\|_\infty = 4$
L_2
L_1
$(0,0)$

Norms $\longrightarrow$ Distances

$$d_p(v, x) = \|v - x\|_p \qquad v, x \in \mathbb{R}^d$$



Norms for Matrices

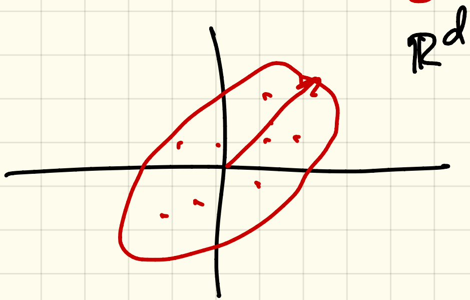
$A \in \mathbb{R}^{n\times d}$

Spectral norm

$$\|A\|_2 = \max_{\substack{x\in\mathbb{R}^d \\ x\neq 0}} \frac{\|\overbrace{Ax}^{\in\mathbb{R}^n}\|_2}{\|x\|_2} = \max_{\substack{y\in\mathbb{R}^n \\ y\neq 0}} \frac{\|\overbrace{yA}^{\in\mathbb{R}^d}\|_2}{\|y\|_2}$$

takes out scale of x, y

$$Ax = \begin{bmatrix} \langle a_1, x\rangle \\ \langle a_2, x\rangle \\ \vdots \\ \langle a_n, x\rangle \end{bmatrix} \in \mathbb{R}^n$$



Frobenius norm

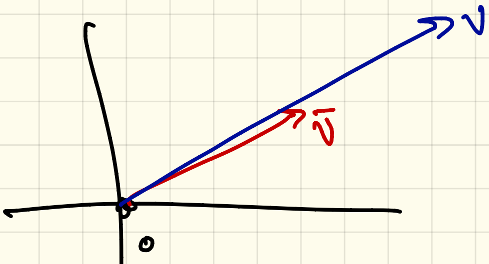
$$\|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{d} A_{ij}^2}$$

unit vector

$v \in \mathbb{R}^d$

so

$\|v\|_2 = 1$



$\bar{v} = \dfrac{v}{\|v\|_2}$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\|A\|_F = \sqrt{1^2 + 2^2 + 3^2 + 4^2}$$

$$= \sqrt{1 + 4 + 9 + 16}$$

$$= \sqrt{30}$$

Linear Independence

set of k vectors $X = \{x_1, x_2, \dots x_k\} \in \mathbb{R}^d$

k scalars $\alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$

$$z = \sum_{i=1}^{k} \alpha_i x_i \in \mathbb{R}^d$$

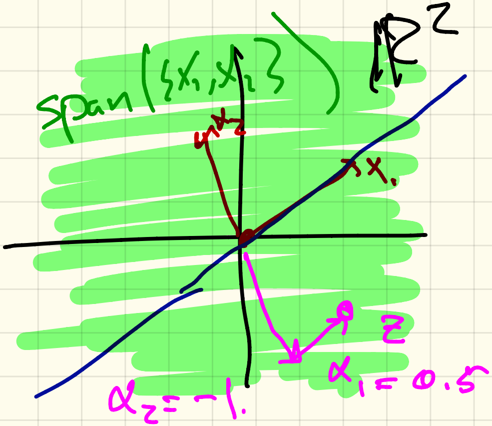
Any vector $z \in \mathbb{R}^d$ that can be written like this linearly <u>dependent</u> on X.

Any vector $z \in \mathbb{R}^d$ that <u>cannot</u> be written $z = \sum_i \alpha_i x_i$ is <u>linearly independent</u>

$$\text{span}(X) = \left\{ z \in \mathbb{R} \;\middle|\; z = \sum_{i=1}^{k} \alpha_i x_i \,,\ \alpha_i \in \mathbb{R} \right\}$$

if $\text{span}(X) = \mathbb{R}^d$ $\quad X \subset \mathbb{R}^d$

then X is a <u>basis</u> of $\mathbb{R}^d$



$$X = \{x_1, x_2\} \qquad x_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \in \mathbb{R}^3$$

$$z_1 = \begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix} \qquad z_2 = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$$

$$z_1 = \overset{\alpha_1}{1}\, x_1 + \overset{\alpha_2}{(-2)}\, x_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix} = z_1$$

$$z_2 = (1)\, x_1 + (1)\, x_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

set of vectors $X = \{x_1 \dots x_n\}$ is

linearly independent

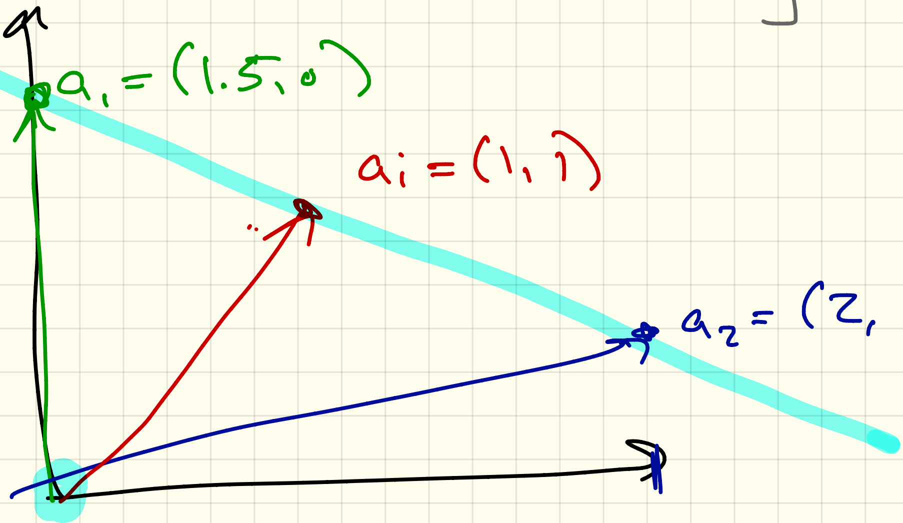
if for all $x_i \in X$

there is no $\{\alpha_1 \dots \alpha_{i-1}, \alpha_{i+1}, \dots \alpha_n\}$

so $x_i = \sum_{\substack{j=1 \\ i \neq j}}^{n} \alpha_j x_j$

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \left[\begin{array}{cc|c} 1.5 & 0 & 0 \\ 2 & 0.5 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

3 pts in $\mathbb{R}^2$



not linearly independent

Rank

Rank of matrix $A \in \mathbb{R}^{n \times d}$

is maximum number of

linearly independent

rows or columns

$$\operatorname{rank}(A) \le \min\{n, d\}$$

$$\operatorname{rank}(A) = \min\{n, d\} \Longrightarrow \text{full rank}$$