FoDA L10

Linear Algebra Review #3

Square Matrices

Rank

Set of vectors $X = \{x_1, \dots, x_n\} \in \mathbb{R}^d$

rank size of the largest subset $X' \subset X$ linearly independent.

$$z = \sum_{i=1}^{k} \alpha_i x_i$$

Rank of Matrix $A \in \mathbb{R}^{n \times d}$ $\quad$ $\text{Rank}(X) = 2$

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_d \\ | & & | \end{bmatrix}$$

rank of rows or of columns
= same =

Matrix $A \in \mathbb{R}^{n \times d}$

$$\operatorname{rank}(A) \leq \min\{n, d\}$$

if $\operatorname{rank}(A) = \min\{n, d\}$

$\hookrightarrow$ "full rank"

Square Matrices

$$A \in \mathbb{R}^{n \times n}$$

$$\boxed{A}$$

Inverse

A^{-1} s.t.

division

$A^{-1} \in \mathbb{R}^{n \times n}$

$$A^{-1}A = I = AA^{-1}$$

$$I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & 1 \\ 0 & & & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

scalar $\alpha \in \mathbb{R}$

$$\alpha^{-1} = \frac{1}{\alpha}$$

$$\cancel{\alpha} \cdot \frac{1}{\cancel{\alpha}} = 1$$

Invertable

A square, full rank

$\text{rank}(A) = n$

Eigenvectors & eigenvalues

v eigenvector $\in \mathbb{R}^n$

λ eigenvalue $\in \mathbb{R}$ (or complex)

square matrix $M \in \mathbb{R}^{n \times n}$

satisfy $M v = \lambda v$

---

if M is full rank

↳ n eigenvector, value pairs

s.t. $\|v\| = 1$

\# eig vec, val $=$ rank(M)

eigenvalues λ might be negative, complex

Postive Definite Matrices

square matrices $M \in \mathbb{R}^{n \times n}$

s.t. have n real positive eigenvalues

for any vector $x \in \mathbb{R}^n$

$$x^T(Mx) > 0$$

Positive Semi Definite Matrix

n real eigenvalues (positive or 0)

any x $\quad x^T M x \geq 0$

$$M = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 5 & 2 \\ 4 & 2 & 8 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

first eigen vector $v_1 = \begin{pmatrix} 0.43 \\ 0.44 \\ 0.78 \end{pmatrix}$ $\lambda_1 = 11.36 = \|M\|_2$

$v_2 = \begin{bmatrix} -0.11 \\ -0.83 \\ 0.54 \end{bmatrix}$ $\lambda_2 = 4.10$

$v_3 = \begin{bmatrix} -0.90 \\ 0.31 \\ 0.31 \end{bmatrix}$ $\lambda_3 = -0.46$

not p.d.

Determinant ↳ square matrix

$$A \in \mathbb{R}^{n \times n}$$

$$|A| \qquad \breve{A}_{ij} \in \mathbb{R}^{n-1 \times n-1} \doteq A \text{ minus } i\text{th row}$$

jth column

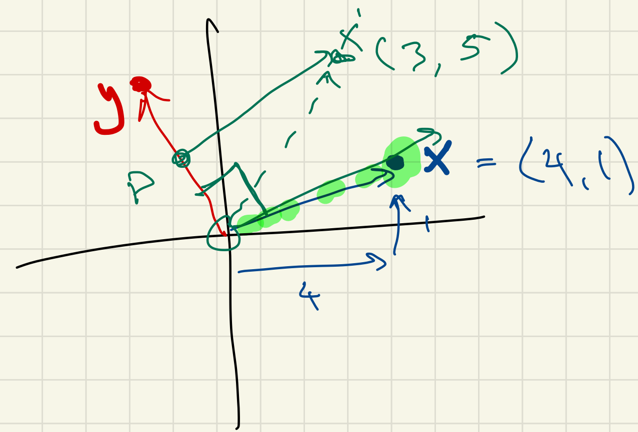
$$|A| = \sum_{i=1}^{n} (-1)^{i+1} A_{1,i} \cdot |\breve{A}_{1,i}|$$

Orthogonality

Two vectors $x, y \in \mathbb{R}^d$

orthogonal if $\langle x, y \rangle = 0$

$$\sum_{j=1}^{d} x_j y_j = 0$$



$x' - P$

Set vectors $v_1 \dots v_k$ → basis

$x = \sum_{i=1}^{n} \alpha_i v_i$

$\|x\| = \sum_{i=1}^{n} \alpha_i^2$

orthonormal

if $\|v_i\| = 1$ all $j \neq i$

$\langle v_i, v_i \rangle = 1$

$\langle v_i, v_j \rangle = 0$

$\langle v_i, v_j \rangle = 0$

for any $x \in \mathbb{R}^n$

Matrix $U \in \mathbb{R}^{n \times n}$ w/ all rows orthonormal and all columns "

$\|x^T U\| = \|x\|$

$UU^T = I$

↳ <u>orthogonal</u>