FoDA L19

Singular Value Decomposition (SVD)

↳ rank-k approximations

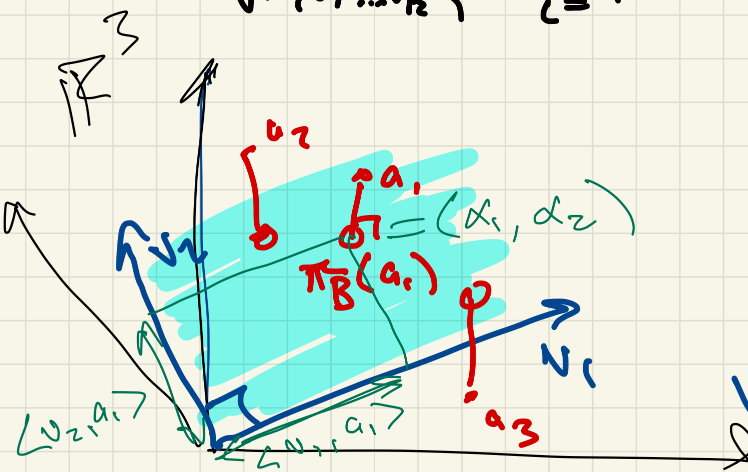
Nov 1, 2022

Input $A \in \mathbb{R}^{n \times d}$ $\quad A = \{a_1, a_2, \dots a_n\}$ $\quad a_i \in \mathbb{R}^d$

Goal Basis B $\quad \boxed{V_B = \{v_1, \dots, v_k\}}$ $\quad k < d$

$v_j \perp v_{j'}$ $\quad \langle v_j, v_{j'} \rangle = 0$ $\quad v_j \in \mathbb{R}^d$ $\quad \|v_j\| = 1$

$$V_B^* = \underset{V = \{v_1, \dots v_k\}}{\arg\min} \sum_{i=1}^{n} \|a_i - \pi_B(a_i)\|^2$$



$\alpha_j = \langle v_j, a \rangle$

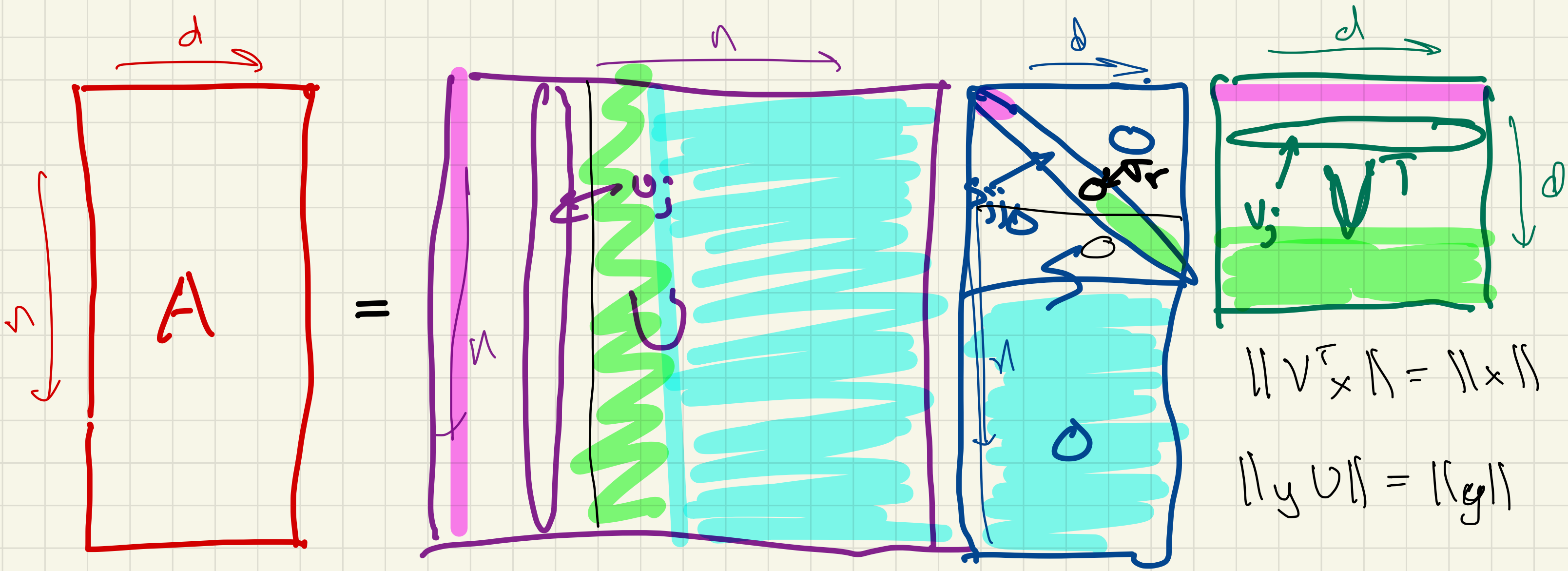
residual $r_i = a_i - \pi_B(a_i)$

$$\pi_B(a_1) = v_1 \underbrace{\langle v_1, a_1 \rangle}_{\alpha_1} + v_2 \underbrace{\langle v_2, a_1 \rangle}_{\alpha_2}$$

$V_B = \{v_1, v_2\}$ $\quad \in \mathbb{R}^d$

$(\alpha_1, \alpha_2) \in \mathbb{R}^2$

Singular Value Decomposition



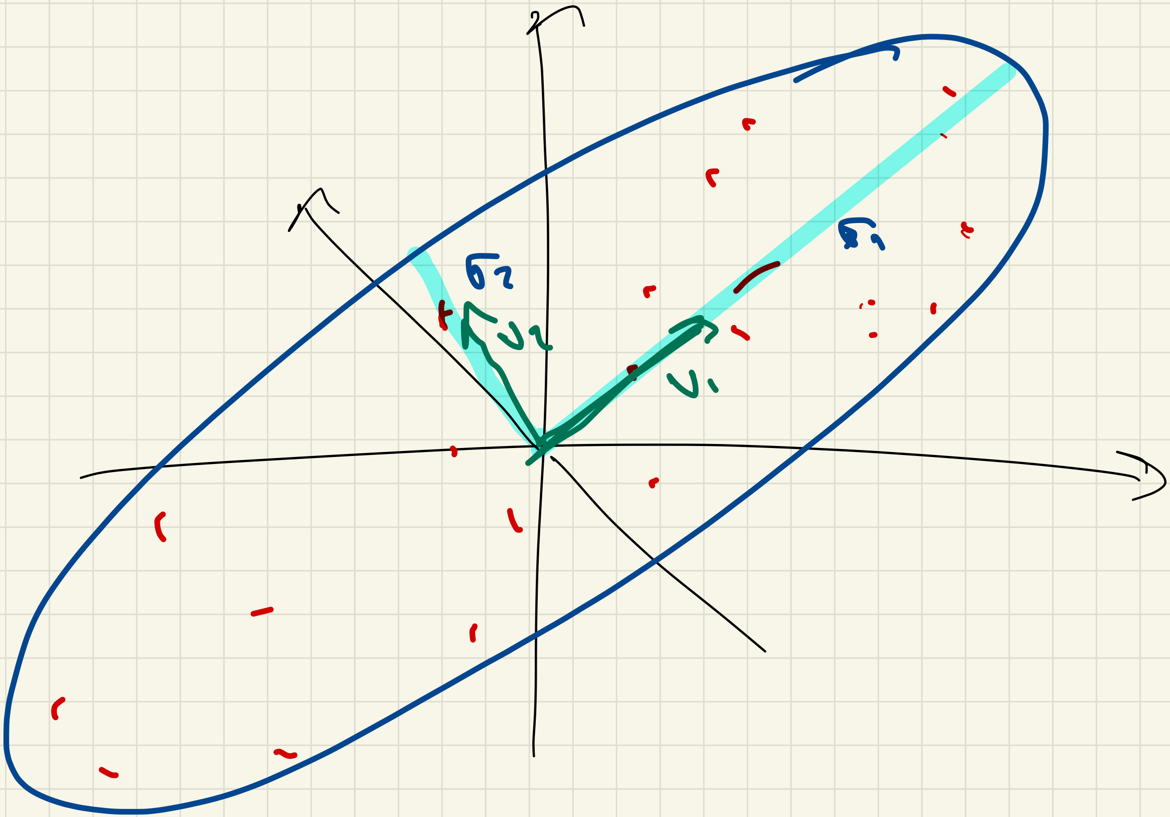
$\|V^T x\| = \|x\|$

$\|yU\| = \|y\|$

U, V orthogonal } left, right singular vectors of A
$u_1 \dots u_n$ $\quad v_1, \dots v_d$

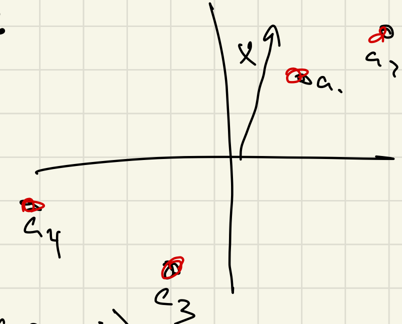
$S = \{S_{11} = \sigma_1, S_{22} = \sigma_2, \dots\}$ singular values

sorted: $\sigma_1 \ge \sigma_2 \ge \dots \ge \sigma_r \ge 0$ $\qquad \sigma_j = 0 \quad j > r = \text{rank}(A)$


σ_2
v_2
σ_1
v_1

$$S = \begin{pmatrix} 8.1655 & 0 \\ 0 & 2.3074 \\ \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & 3 \\ 2 & 2 \\ -1 & -3 \\ -5 & -2 \end{pmatrix} \in \mathbb{R}^{4 \times 2}$$



$$x \approx (0.243, 0.970)$$

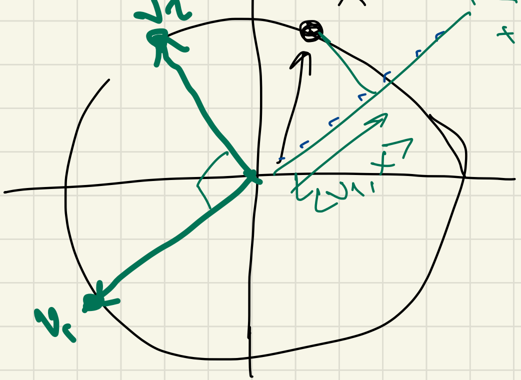
$$V = \begin{array}{cc} \begin{pmatrix} -0.8142 & -0.5805 \\ -0.5805 & 0.8142 \end{pmatrix} \\ \quad v_1 \qquad\qquad v_2 \end{array}$$

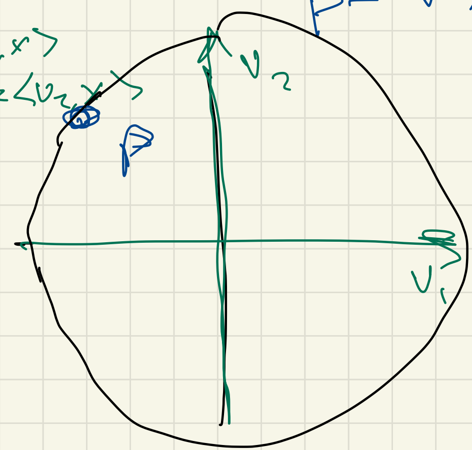
$$Ax = U(S(V^T x))$$

$$p = V^T x = (\langle v_1, x \rangle, \langle v_2, x \rangle)$$

$$x = v_1 \langle v_1, x \rangle + v_2 \langle v_2, x \rangle$$

$$q = S V^T x = S p$$



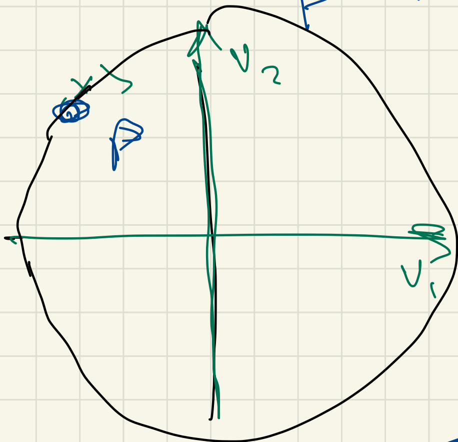


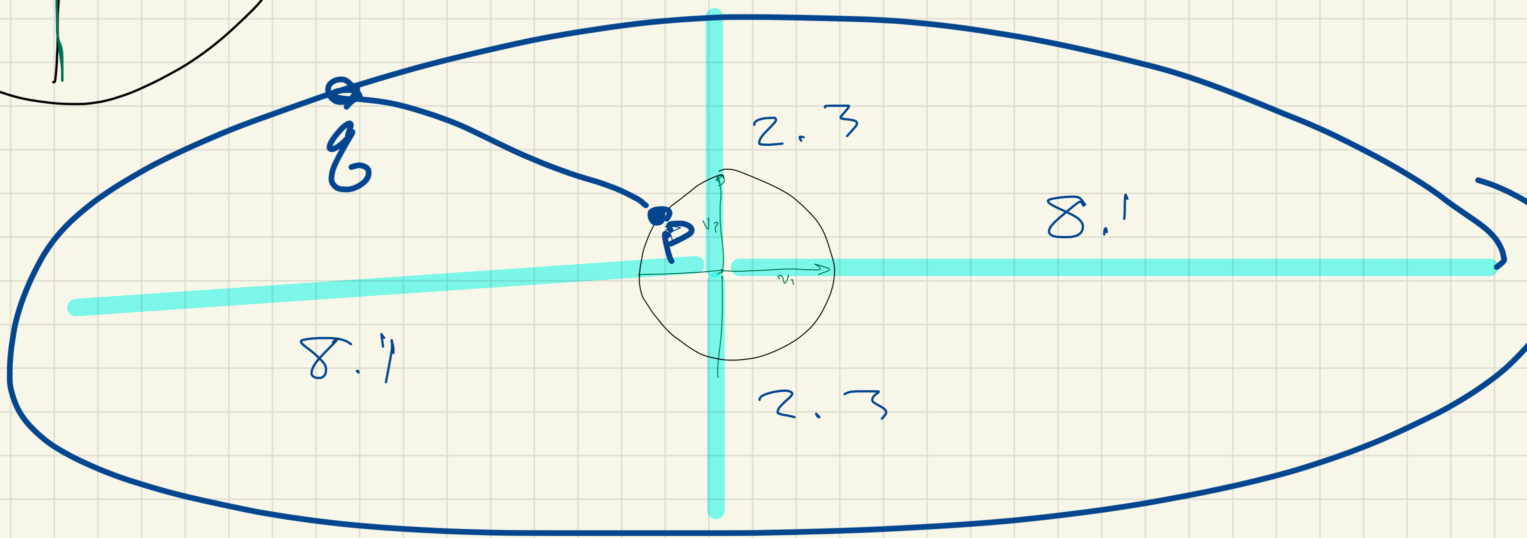
$$p = V^T x = (\langle v_1, x\rangle, \langle v_2, x\rangle)$$

$$q = \underline{\Sigma V^T x} = \Sigma p$$

$$y = Ax$$

$$y = U \Sigma V^T x \in \mathbb{R}^4$$





Input $A \in \mathbb{R}^{n \times d}$

$svd(A) = U S V^T$ $\qquad V = \{v_1, v_2, \dots v_d\}$

$$\{v_1, \dots v_k\} = \underset{\substack{B \\ rank(B)=k}}{\operatorname{argmin}} \sum_{i=1}^{n} \| a_i - \pi_B(a_i) \|^2$$

Σ first k right sing. vectors

Note: given that B contains $\underline{0} = (0, 0 \dots 0)$

What is? $\sum_{i=1}^{n} \| a_i - \pi_{v_1, \dots v_k}(a_i) \|^2$

$$= \sum_{j=k+1}^{d} \sigma_j^2$$

$\leftarrow$ sum of sqrd sing. values rounded to ∞.

Best Rank-k approximation of A

↳ A_k

$$A_k = \underset{A' \text{ rank } k}{\arg\min} \|A' - A\|^2_{F \text{ or } 2}$$

$$A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^T$$

$$A_k = U_k \, S_k \, V_k^T$$

$$U_k = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_k \\ | & | & & | \end{bmatrix}$$

$$S_k = \begin{bmatrix} \sigma_1 \cdots \sigma_k & 0 \\ 0 & \end{bmatrix}$$

$$V_k^T = \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_k & - \end{bmatrix}$$