FoDA L7

Convergence

PAC & Concentration of Measure

Sep 13, 2023

Quiz #1

Average Score 89%

Average Time 25 min.

goal [15–20 min]

R.V. X equation $Y = \alpha X + c$ (const.)

$$E[Y] = \alpha E[X] + c$$

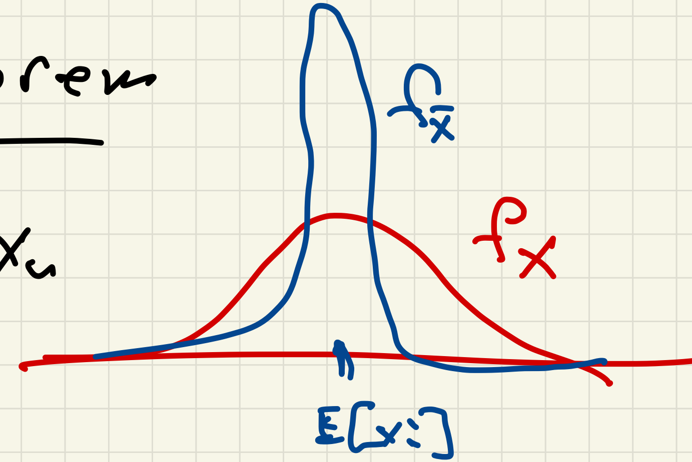
$$\mathrm{var}[Y] = \alpha^2 \mathrm{var}[X]$$

Central Limit Theorem

n iid R.V. $X_1, \ldots, X_n$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- $E[\bar{X}] = E[X_i]$
- $Var[\bar{X}] = \dfrac{Var[X_i]}{n}$
- $\bar{X}$ looks "normal"



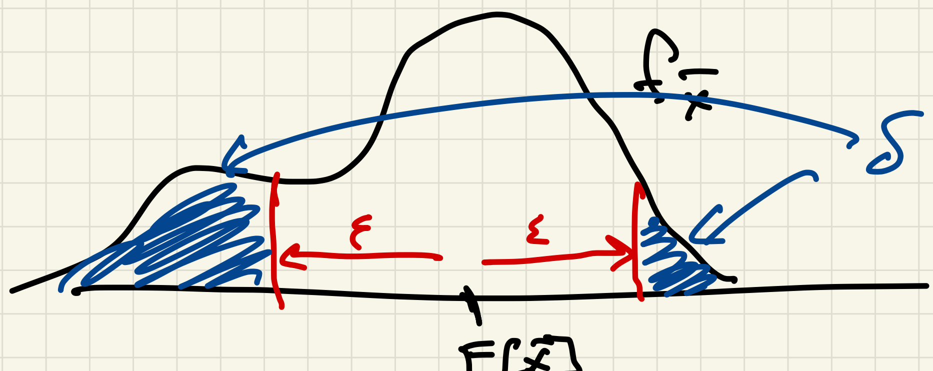
PAC

How accurate is $\bar{X}$?

$$\Pr\left[\left| \bar{x} - E[\bar{x}] \right| > \varepsilon \right] < \delta$$

probability of failure

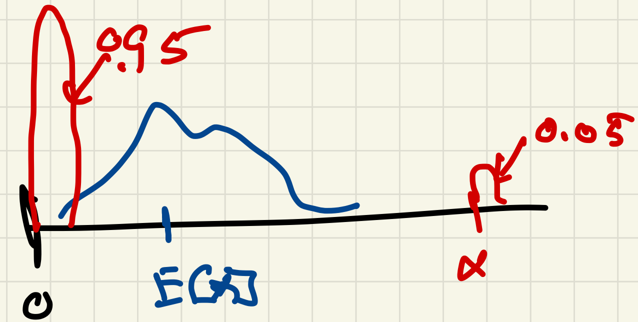
error tolerance



Markov Inequality

R.V. X

- $E[X]$
- $X \geq 0$



for any $\alpha > 0$

$$\Pr[X \geq \alpha] \leq \frac{E[X]}{\alpha} = \delta$$

$$\Pr(X - E[X] \geq \epsilon] \leq \frac{E[X]}{\alpha - E[X]}$$

$$\epsilon := \alpha - E(X)$$

$$E[X] = 0 \cdot \underbrace{\Pr[X = 0]}_{1-\delta} + \alpha \cdot \underbrace{\Pr[X = \alpha]}_{\delta}$$

$$= 0 + \alpha\delta$$

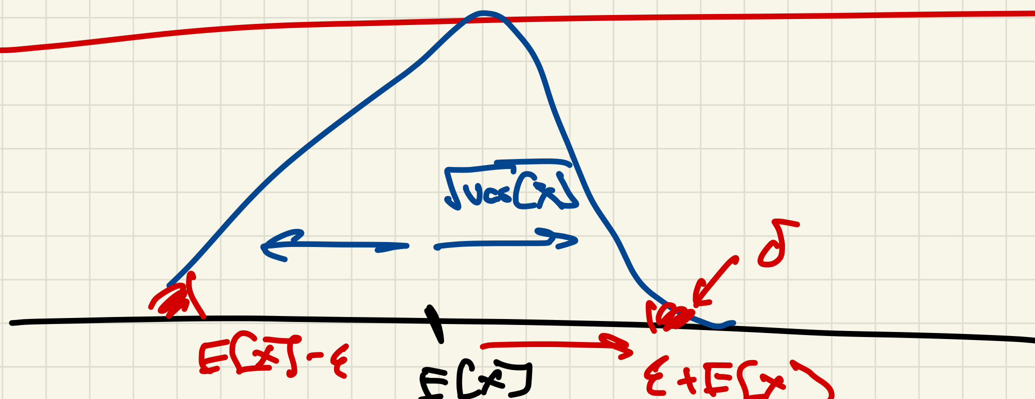
$$\frac{E[X]}{\alpha} = \delta$$

Chebyshev Inequality

R.V. X

- $E[X]$
- $Var[X]$

$$\boxed{\Pr\left[\,|X - E[X]| \geq \varepsilon\,\right] \leq \frac{Var[X]}{\varepsilon^2} = \delta}$$



R.V. R = rain SLC in June

- $R > 0$
- $E[R] = 20$ mm

$$\Pr[R \geq 50\text{mm}] \leq \frac{E[R]}{50\text{mm}} = \frac{20\text{mm}}{50\text{mm}} = 0.4$$

Markov

- $\text{Var}[R] = 9 \text{ mm}^2$

$$\Pr[R \geq 50\text{mm}] = \Pr[R - E[R] \geq 50\text{m} - E[R]]$$

$$\leq \Pr[|R - E[R]| \geq 30\text{mm}] \leq \frac{\text{Var}[R]}{(30\text{ mm})^2} = \frac{9\text{ mm}^2}{900\text{ mm}^2} = \frac{1}{100}$$

$$\Pr[R > 50\text{m}] \leq \min\{0.4, 0.01\}$$

Chebyshev

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \quad \text{iid}$$

$$Var[X_i] = \sigma^2$$

$$Var[\bar{X}] = \sigma^2 / n$$

$$\Pr\{|\bar{X} - E[\bar{X}]| \geq \varepsilon\} \leq \frac{Var[\bar{X}]}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2 n} = \delta$$

solve for n

ε - error tolerance

δ = prob. failure

σ^2 ← variance

$$n = \frac{\sigma^2}{\varepsilon^2 \delta}$$

Chernoff - Hoeffding Ineq.

R.V.s $X_1, X_2, \ldots X_n$ iid f_X

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- $E[X_i]$
- $X_i \in [a, b]$ $\quad b - a = \Delta$

2.71…

$\exp(z) = e^z$

$\varepsilon > 0$

$$\Pr\left[\,|\bar{X} - E[\bar{X}]| > \varepsilon\,\right] \leq \underbrace{2\exp\left(-\frac{2\varepsilon^2 n}{\Delta^2}\right)}_{=\delta}$$

solve for n

$$n = \frac{\Delta^2}{2\varepsilon^2}\ln\left(\frac{2}{\delta}\right)$$

Die $= \{1 \ldots 6\}$ $\quad n = 120$ rolls

$$T_i = \begin{cases} 1 & \text{if } 3 \\ 0 & \text{o.w.} \end{cases} \qquad T = \# \, 3\text{s} \qquad \bar{T} = \text{fraction of 3's}$$

$\Delta = 1$

$1/6$ $\qquad E[T] = 20 \qquad \bar{T} = \frac{T}{120}$

$$\Pr[T > 40] \leq \Pr\left[|\bar{T} - E[\bar{T}]| \geq \frac{1}{6}\right]$$

$$\leq 2\exp\left(-\frac{2(1/6)^2 \cdot 120}{1^2 = \Delta^2}\right)$$

$$- \frac{2 \cdot 120}{36} = \frac{-20}{3}$$

$$= 2\exp\left(\frac{-20}{3}\right) \leq 0.0026$$

Chebyshev $\quad var[T_i] = 5/36$

$$\Pr[T > 40] \leq \Pr\left[|\bar{T} - E[\bar{T}]| \geq 1/6\right] \leq \frac{Var[\bar{T}]}{n \cdot (1/6)^2} = \frac{5/36}{120 \cdot 1/6^2}$$

$$\leq 0.42$$

$$\frac{T}{120} > \frac{40}{120} \qquad \bar{T} = \frac{T}{n} = \frac{T}{120}$$

$$\bar{T} - E[\bar{T}] > \frac{1}{3} - E[\bar{T}] \qquad E[\bar{T}] = \frac{1}{6} \qquad \Delta \overset{?}{=} 1$$

$$\bar{T} - \frac{1}{6} > \frac{1}{3} - \frac{1}{6}$$

$$\Pr\left[\bar{T} - E[\bar{T}] \geq \frac{1}{3} - \frac{1}{6} = \frac{1}{6}\right]$$

$$\leq$$

$$\Pr\left[\,\left|\bar{T} - E[\bar{T}]\right| \geq \frac{1}{6}\right]$$