Yes, I recorded some lectures.

L18: Compressed Sensing and Orthogonal Matching Pursuit

Jeff M. Phillips

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Sparse Sensing

$S \in \mathbb{R}^d$ $d = 32$

only m 1 bits $m = 8$

(in general $m \ll d$)

"Sparse" Signal

choose random

$$S^T = [0\,1\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,1\,1\,0\,1\,0\,0\,0\,1\,0\,0\,1\,0\,0]$$

$$x_i^T = [\text{-}1\,0\,1\,0\,1\,1\,\text{-}1\,1\,0\,\text{-}1\,0\,0\,1\,\text{-}1\,\text{-}1\,1\,0\,1\,0\,1\,\text{-}1\,\text{-}1\,\text{-}1\,0\,1\,0\,0\,\text{-}1\,0\,1\,0\,0]$$

random measurement

$$y_i = \langle S, x_i \rangle$$

random aggregated view of S.

$$= 0{+}0{+}0{+}0{+}1{+}0{+}0{+}0{+}0{+}0{+}0{+}0{+}0{+}0{+}0{+}1{+}0{+}0{+}0{+}1{-}1{+}0{-}1{+}0{+}0{+}0{+}0{+}0{+}0{+}1{+}0{+}0$$

$$= 2$$

<u>Goal:</u> Find 1 bits in S.

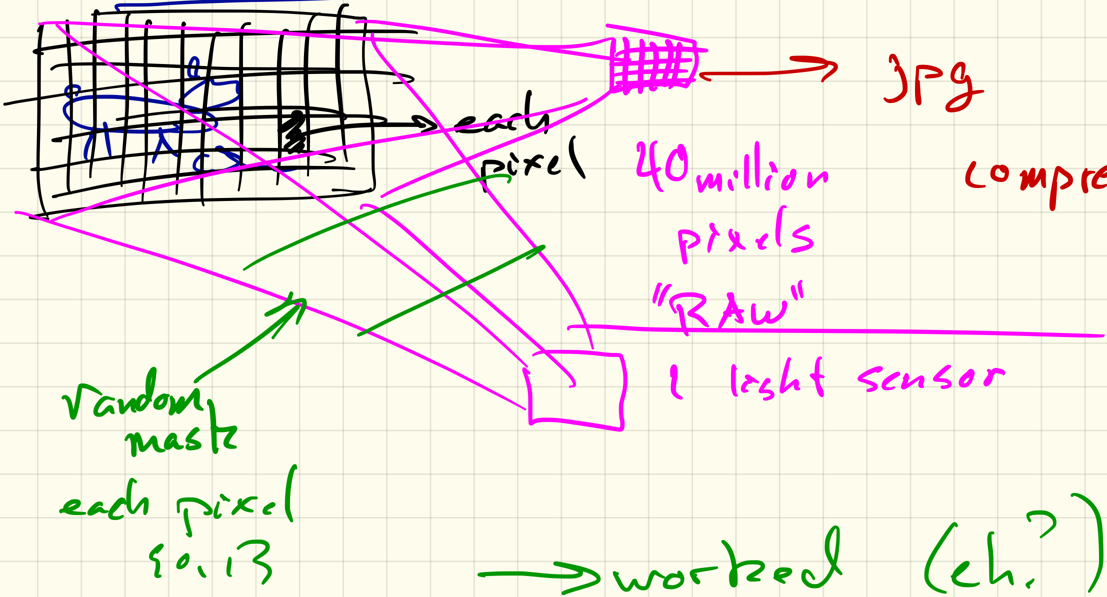
<u>Claim</u>: Recover S <u>exactly</u> with only $N = K \cdot m \lg(d/m)$ measurements.

$x_{ij} \in \mathrm{Unif}\{-1, 0, +1\}$

$x_i \in \{-1, 0, +1\}^d$

Examples of Compressed Sensing

- Single Pixel Camera



→ worked (eh?) o.k.

- Hubble Telescope // • MRI on kids.

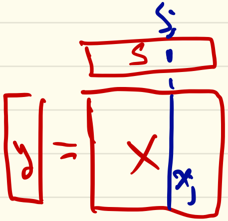
Orthogonal Matching Pursuit (OMP)

Input Measurement Matrix X

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$N = K \cdot m \log(d/m)$

$x_i \in \{-1, 0, +1\}^d$



$y = Xs$ $\qquad y \in \mathbb{R}^N$

- Choose column $X_j = \underset{X_{j'} \in X}{\arg\max} \left|\langle y, X_{j'} \rangle\right|$
- Estimate $\gamma = \underset{\gamma \in \mathbb{R} \text{ or } \{0,1\}}{\arg\min} \|y - X_j \gamma\|$ $\quad (\gamma \approx s_j)$
- Update $y = y - X_j \gamma$ $\qquad$ Set $s_j = \gamma$

Orthogonal Matching Pursuit (OMP)

Orthogonal Matching Pursuit

Set $r = y$.

for $i = 1$ **to** t **do**

Set $X_j = \arg\max_{X_{j'} \in X} |\langle r, X_{j'} \rangle|$.

Set $\gamma_j = \arg\min_\gamma \|r - X_j \gamma\|$.

Set $r = r - X_j \gamma_j$.

Return $\hat{S}$ where $\hat{s}_j = \gamma_j$ (or 0).

r residual

OMP Example

$d = 10$ $m = 3$

signal: $S = [0, 0, 1, 0, 0, 1, 0, 0, 1, 0]$

X_6 X_7 X_8

$$\text{measurement:} \quad X = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 & -1 & 1 & -1 & 0 \end{bmatrix}$$

0 0 1 2 3 1 2 3 4 0

0 1 2 0 0 0 0 1 0 0

0 1 2 1 0 2 2 2 1 1

observation: $y = XS^T = [0, 0, 0, 1, 1, -2]^T$

$|\langle y, X_1 \rangle| = 0$

$X_9 = \underset{X_{j'}}{\arg\max} |\langle y, X_{j'} \rangle|$

$r_1 = y - X_{j'} \cdot 1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$

Set $S_9 = 1$

$X_3 = \underset{X_{j'}}{\arg\max} |\langle r_1, X_{j'} \rangle|$

$r_2 = r_1 - X_3 \cdot 1 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Set $S_3 = 1$

$\{X_3, X_6, X_7, X_8\} = \underset{X_{j'}}{\arg\max} |\langle r_2, X_{j'} \rangle|$

$r_3 = r_2 - X_6 \cdot 1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$r_3 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ implies

$y = X_9 + X_3 + X_6$ (+Noise)

OMP Example

signal: $S = [0, 0, 1, 0, 0, 1, 0, 0, 1, 0]$

measurement: $$X = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 & -1 & 1 & -1 & 0 \end{bmatrix}$$

observation: $y = XS^T = [0, 0, 0, 1, 1, -2]^T$

OMP Example

signal: $S = [0, 0, 1, 0, 0, 1, 0, 0, 1, 0]$

measurement: $$X = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 & -1 & 1 & -1 & 0 \end{bmatrix}$$

observation: $y = XS^T = [0, 0, 0, 1, 1, -2]^T$

OMP Example

signal: $S = [0, 0, 1, 0, 0, 1, 0, 0, 1, 0]$

measurement: $$X = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 & -1 & 1 & -1 & 0 \end{bmatrix}$$

observation: $y = XS^T = [0, 0, 0, 1, 1, -2]^T$

$$E\left[\langle y, X_j \rangle\right] = \sum_{i=1}^{d} E\left[y_i \cdot X_{j_i}\right]$$

s.t. $S_j = 0$

$$= \sum_{i=1}^{d} y_i \, E\left[X_{j_i}\right]$$

$$= \sum_{i=1}^{d} y_i \cdot 0 = 0$$

Optimizations

① $S_j = \underset{\gamma}{\operatorname{argmin}} \| r - X_j \cdot \gamma \| + \alpha |\gamma|$

② $[\gamma_1, \gamma_2 \ldots \gamma_t] = \underset{[\gamma_1 \gamma_2 \ldots \gamma_t]}{\operatorname{argmin}} \| y - \sum_{s=1}^{t} X_{j_s} \cdot \gamma_{j_s} \| + \alpha \|\gamma\|_1$

bias towards sparse S.

instead ① $\alpha \|\gamma\|_2$

② Ridge Reg

③ Round γ_{j_s} to 0 if small enough

MG

$$\hat{f}_\xi \in [f_\xi - \varepsilon n, f_\xi]$$

$$f_\xi - \varepsilon n \leq \hat{f}_\xi \leq f_\xi$$

CM (w.p. $\geq 1-\delta$)

$$f_\xi \leq \hat{f}_\xi \leq f_\xi + \varepsilon n$$

$$\hat{f}_\xi \in [f_\xi, f_\xi + \varepsilon n]$$

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